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APPLICATION OF A GRADIENT APPROACH TO ESTIMATION OF THE LOCAL STRENGTH

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Some problems associated with the use of the gradient approach for estimating the local strength are considered. It is shown that a physically unjustified choice of the gradient function in the strength criterion can lead to contradictory results.

The gradient criteria of the ultimate state have recently been developed intensively [1-6]. Both the general approaches and the particular local strength and yield criteria have been elaborated. As a whole, the gradient criteria describe well the occurrence of an ultimate state in local regions, in particular, in stress-concentration zones. However, in some cases, their use give rise to contradictory results. Here we show that the correct application of the gradient approach to estimation of the local strength allows one to obtain physically correct expressions for the ultimate stresses and dimensions of the defects.

The traditional approach to strength calculations is to compare the internal stresses which occur in a deformable body with their limiting values. The strength condition has the form

$$\sigma_e \leqslant \sigma_0, \tag{1}$$

where $\sigma_e = f(\sigma_{ij})$ and $\sigma_0 = \text{const}$. The equivalent stress σ_e characterizes the internal intense stress state of the body and is a function of the stress-tensor components σ_{ij} in the general case. The ultimate stress σ_0 characterizes the average mechanical properties of the body's field and it is assumed to be a constant of the material. Since σ_0 is determined for the uniform stress state, the range of application of the traditional approach is restricted to the cases where the dimension of the stress-uniformity zone is quite large to consider that $\sigma_0 = \text{const}$. Therefore, the gradient approach is used to estimate the local strength. In contrast to the traditional approach, the essence of the gradient approach is to assign the mechanical properties to a certain deformable region of finite dimensions rather than to the material as such, which is more appropriate for the concept of mechanical strength. This means that the ultimate stress is not a constant of the material and depends on the dimensions of the stress-uniformity zone.

The characteristic dimension of the deformable region is denoted by L_e; if it is quite large compared to the dimensions of the structural components of the material, including the admissible defects of the structure, i.e., the conditions of averaging of the mechanical properties are satisfied, the value of the local strength differs little from σ_0. On the contrary, if L_e is comparable with the dimensions of the structural units, their influence on the local strength becomes noticeable. This influence is the stronger, the smaller the dimension L_e relative to the characteristic dimension of the structure of the material L_0. Thus, the local strength of the material should depend not only on the characteristic dimension of the deformable region L_e but also on the ratio L_0/L_e. With allowance for this, we write the local-strength condition

$$\sigma_e \leqslant f(\sigma_0, L_0/L_e). \tag{2}$$

It is hardly possible to find a universal form of the function $f(\sigma_0, L_0/L_e)$; however, one can formulate additional conditions which reflect the specifics of the problem in each particular case and to which this

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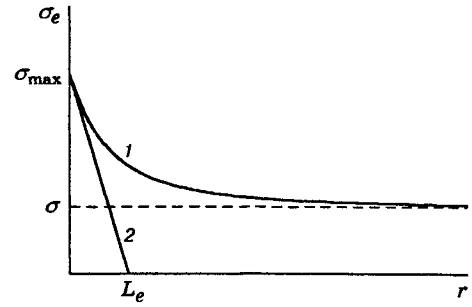

Fig. 1

function should correspond. The form of $f(\sigma_0, L_0/L_e)$ is determined with account of these conditions. Let us formulate the requirements for $f(\sigma_0, L_0/L_e)$ for the problem of stress concentration:

— Taking into account the stress gradients (the gradient hypothesis)

$$f(\sigma_0, L_0/L_e) = \sigma_0[1 + f^*(\sigma_{ij}, \sigma_{ij,k})], \qquad f^*(\sigma_{ij}, \sigma_{ij,k}) = \text{inv}; \tag{3}$$

— Link to the traditional criteria

$$f(\sigma_0, 0) = \sigma_0; \tag{4}$$

— Limitedness of the critical stresses

$$f(\sigma_0, L_0/L_e)/K_t \to \text{const}, \qquad K_t \to \infty. \tag{5}$$

Here $f^*(\sigma_{ij}, \sigma_{ij,k})$ is a dimensionless function of the components of the stress tensor and the stress-gradient tensor which is invariant under coordinate transformations and K_t is the stress-concentration coefficient.

The requirement (3) is due to the fact that in the stress-concentration conditions the characteristic dimension of the deformable region is determined by the dimensions of the stress-uniformity zones, rather than the dimension of the whole body. This dimension depends on the character of the stress distribution and, hence, their gradients. The gradient hypothesis was formulated by Serensen [7] [in a form different from (3)] and was used by Strelyaev, Afanas'ev, et al. to describe the experimental brittle static and fatigue strength data of the specimens with geometrical stress concentrators (orifices, notches, and hollow chamfers). In accordance with the gradient hypothesis, the onset of the ultimate state is determined by the values of the stresses themselves and their gradients at a given point. The use of the gradient hypothesis is a key factor in the development of gradient strength criteria [3, 4]; however, this has meaning only if the stress gradients do not vanish at dangerous points (for example, by virtue of symmetry of the problem).

The requirement (4) ensures the transition of the gradient (2) to the traditional (1) criterion in the case of a uniform stress state.

The requirement (5) reflects the known experimental fact: irrespective of the degree of acuteness of a notch and the value of the theoretical stress-concentration coefficient, the body fails under a finite load [8]. For acute concentrators, its value depends only on the length of a cut. The requirement (5) ensures as a matter of fact the relation between the gradient criterion (2) and linear fracture mechanics.

With allowance for the requirements (3)–(5), we propose the following gradient criterion of the ultimate state [1, 2]:

$$\sigma_e \leqslant \sigma_0(1 + \sqrt{L_0/L_e}); \tag{6}$$

$$L_e = \sigma_e/|\text{grad}\,\sigma_e|. \tag{7}$$

The characteristic dimension of the deformable region determined by expression (7) is shown in Fig. 1. Curve 1 characterizes the equivalent-stress distribution over the dangerous cut, the straight line 2 is tangent to curve 1 at the point of stress concentration ($\sigma_e = \sigma_{max}$), and the dashed straight line is the asymptote.

The criterion (6) describes quite precisely the experimental data on the onset of a local plastic flow in the stress-concentration region, which were obtained for some metal materials on flat specimens with orifices and notches of various forms [5].

Legan [3] suggested another method of estimating the local strength. In addition to the function of the criterion (6)

$$f(\sigma_0, L_0/L_e) = \sigma_0(1 + \sqrt{L_0/L_e}), \tag{8}$$

the function

$$f(\sigma_0, L_0/L_e) = \sigma_0\sqrt{1 + L_0/L_e} \tag{9}$$

and some combinations of (8) and (9) satisfy the requirements (3)–(5). In view of this, we consider the gradient criterion

$$\sigma_e \leqslant \sigma_0(1 - \beta + \sqrt{\beta^2 + L_0/L_e}), \tag{10}$$

where β is a dimensionless parameter. The characteristic dimension L_e is estimated by (7), and the greatest normal stress is regarded as an equivalent dimension.

The introduction of an additional parameter makes the gradient criterion more flexible and allows one to describe the experimental data more accurately, using β as the approximation parameter. However, not only the function (8) and (9) but many others meet the requirements (3)–(5). The choice of the function $f(\sigma_0, L_0/L_e)$ should be physically justified. The formal complication of the gradient function does not allow one to explain the physical meaning of the newly introduced parameters and leads to absurd results in some cases.

Just as in fracture mechanics, in the framework of the gradient approach one can pose the problem of estimation of the admitted dimensions of defects and determination of their critical values. However, there are no restrictions connected with the type of defect; the latter can have any geometrical form, not precisely a crack-like shape. The admitted dimensions of the defects are determined from the condition

$$\sigma_c \geqslant \sigma_0, \tag{11}$$

where σ_c is the critical value of the applied stress at which the ultimate state is reached at the most stressed point of the body. Condition (11) means that the presence of defects that are not the stress concentrators does not result in a decrease in the ultimate stress compared to the strength σ_0 of a "defectless" material. According to the gradient approach, the critical stress is estimated as follows:

$$\sigma_c = f(\sigma_0, L_0/L_e)/K_t. \tag{12}$$

Using the gradient criterion (10), with allowance for (11) and (12), we obtain an estimate of the critical dimension of the defect in the form of an open elliptic orifice in a plate [3]:

$$l_c = l_0 \frac{K_t - 1}{K_t + 2\beta - 1}\left(1 + \frac{1}{2K_t}\right), \tag{13}$$

where $l_0 = 2K_c^2/(\pi\sigma_0^2)$ is the critical dimension of the defect in the form of a longitudinal crack and K_c is the critical coefficient of stress intensity. It is easy to show that the dependence l_c (13) contradicts the common meaning. For $\beta \geqslant 0.25$, the critical dimension of the defect decreases as the acuteness of the notch decreases, vanishing for $K_t = 1$, i.e., the less acute concentrators become more dangerous owing to the decrease in the critical dimension. In the range $0 < \beta < 0.25$, the value of l_c changes nonmonotonically, the dependence l_c acquires the physical meaning only for $\beta = 0$, and precisely this case corresponds to the use of the criterion (6). Thus, although the introduction of β into the gradient criterion by the method proposed in [3] makes it more flexible from the viewpoint of the description of experimental local strength data, it is physically unjustified.

To eliminate this contradiction, we present l_c in the form

$$l_c = l_0\left(1 + \frac{\beta}{K_t}\right), \qquad \beta \geqslant 0, \tag{14}$$

where β is a numerical parameter. The physically consistent values of β lie in the domain $\beta \geqslant 0$. It is noteworthy that, because the dimension L_e is characteristic, it can be estimated by various methods, not only by means of expressions (7); therefore, the satisfaction of the requirement (3) is not, strictly speaking, obligatory. In solving an elastoplastic problem with the use of the gradient yield condition [5], it is convenient to use the inverse value relative to the stress to determine L_e [5]. However, there is no need in it to estimate the local strength of a linearly elastic body, the more so since the determination of the relative stress gradient is labor-consuming. It is known [9, 10] that the local stress distribution depends on the radius of curvature of the concentrator to a larger extent than on other geometrical parameters; therefore, in the first approximation, one can use the radius of curvature of the concentrator ρ at a dangerous point to estimate L_e. Here the restrictions connected with the use of the gradient hypothesis are eliminated, because it is not necessary to calculate the stress gradients. For estimation of L_0, the critical dimension of the defect l_c is used. We present the function $f(\sigma_0, L_0/L_e)$ in the form

$$f(\sigma_0, L_0/L_e) = \sigma_0 f(l_c/\rho). \tag{15}$$

Bearing in mind that the stress-concentration coefficient is an increasing function of l/ρ (l is the dimension of the concentrator)

$$K_t = f_t(l/\rho), \tag{16}$$

it is easy to see that it suffices to use the function f_t as $f(l_c/\rho)$ to satisfy the requirements (4) and (5):

$$f(l_c/\rho) = f_t(l_c/\rho). \tag{17}$$

Thus, with allowance for (15) and (17), the local strength criterion takes the form

$$\sigma_e \leqslant \sigma_0 f_t(l_c/\rho). \tag{18}$$

Therefore, the ultimate stress is determined by the expression $\sigma_c = \sigma_0 f_t(l_c/\rho)/f_t(l/\rho)$, and the ratio $f_t(l/\rho)/f_t(l_c/\rho)$ can be regarded as an effective stress-concentration coefficient.

For numerous important applied problems, the stress-concentration coefficient can be approximated (or exactly presented) in the form [11]

$$K_t = 1 + \alpha\sqrt{a/\rho}, \tag{19}$$

where α is a numerical coefficient which depends on the geometry of the body and $a = l/2$. For these problems, the local-strength criterion (18) is transformed to the form (6) for $L_0 = \alpha^2 l_c/2$. However, if the requirements imposed on the accuracy of determination of the ultimate stress do not admit this, the gradient function should be determined according to (15) and (17).

We consider some examples. Let us begin with the uniaxial tension of an infinite plane with an elliptic orifice. The problem was solved by Kolosov [12] and Inglis [13]. The stress-concentration coefficient is determined by expression (19) with $\alpha = 2$. The critical stress has the form $\sigma_c = \sigma_0(1 + \sqrt{2l_c/\rho})/K_t$, where l_c is determined according to (14). The parameter β is found experimentally, but it is necessary to set β equal to zero if the necessary experimental data are absent. In this case, the lower estimates of the critical dimension of the concentrator and the ultimate stress, which are assigned to the strength limit, are obtained. The parameter β can also be determined by a different way if one uses expression (7) to estimate the characteristic dimension L_e. In this example, the resulting value of β is equal to 0.5 (if the greatest normal stress is used as the equivalent stress) or 0.4 (if the stress intensity is used) [6]. However, the validity of these estimates can be supported only experimentally.

For a plate of finite dimensions with a central elliptic orifice, the stress-concentration coefficient can also be presented in the form (19). Here the coefficient α depends on the dimensions of the plate and is found

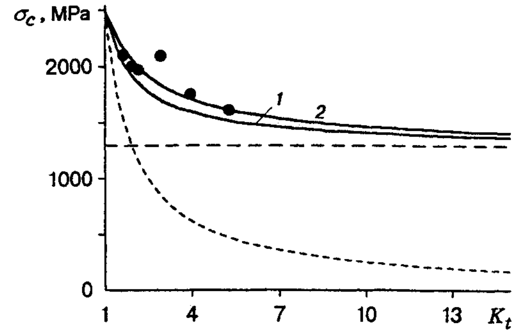

Fig. 2

by known approximate formulas [11]. In this case, the ultimate stress has the form

$$\sigma_c = \sigma_0(1 + \alpha\sqrt{l_c/(2\rho)})/K_t. \tag{20}$$

Here l_c is determined by expression (14), but the quantity l_0 entering into it should be corrected with allowance for the finite dimensions of the plate: $l_0 = (2K_c^2/(\pi\sigma_0^2))(1/F^2)$, where F is a correction function which takes into account the effect of the dimensions of the body on the stress-intensity coefficient [14].

Expressions (19) and (20) are applicable also to concentrators of nonelliptic form, for which one can introduce the notion of an "equivalent elliptic orifice" or "equivalent elliptic notch" [11]. The latter concerns both flat and cylindrical specimens with a surface circular notch, including a V-shaped notch with a small opening angle. Nisitani and Nogushi [15] reported the experimental data concerning the determination of the fracture stress σ_c upon tension of cylindrical specimens having a circular notch. The specimens were fabricated from S45C high-strength steel. The notch was V-shaped with opening angle $\psi = 60°$ and radius of curving ρ at the top. Specimens with notch depth $a = 0.2$ mm were tested by varying ρ within 0.056–2.1 mm. The minimum diameter of the transverse cross section was constant and equal to 4.5 mm.

Figure 2 shows the values of σ_c calculated by formula (20), depending on the stress-concentration coefficient for $\beta = 0$ and $\beta = 1$ (curves 1 and 2). Curve 1 limits from below the domain of σ_c, and curve 2 approximates the experimental data represented by the dots. As $K_t \to \infty$, the calculated curves approach asymptotically the value found in accordance with the linear approach of fracture mechanics (dashed straight line). The dashed curve is calculated according to the traditional approach. The experimental data demonstrate convincingly the advantage of the gradient approach over the standard approach.

Now we consider the uniaxial tension of a unbounded body weakened by an internal cavity, which is the ellipsoid of revolution about the axis of loading. As $K_t \to \infty$, the ellipsoid becomes a round crack of radius a. The stress-concentration coefficient is connected with a and ρ by the following relation [9]:

$$K_t = \frac{1 - \nu - (1.5 - \nu)a/\rho + 2(a/\rho)^2 + (\nu - (1.5 + \nu)a/\rho)a\dot{c}/\rho}{1 - \nu + a/\rho + (a/\rho - 2 + 2\nu)ac/\rho - (1 + \nu)(ac/\rho)^2}. \tag{21}$$

Here

$$c = \frac{\arctan\sqrt{a/\rho - 1}}{\sqrt{a/\rho - 1}} \text{ for } a/\rho > 1, \quad c = \frac{\ln(1 + \sqrt{1 - a/\rho}) - (1/2)\ln(a/\rho)}{\sqrt{1 - a/\rho}} \text{ for } a/\rho < 1,$$

and ν is the Poisson ratio. If $\rho = a$, we have

$$K_t = \frac{3(9 - 5\nu)}{2(7 - 5\nu)}. \tag{22}$$

This problem is of primary interest for evaluating the danger of internal defects. According to the gradient approach, the critical dimension of the defect is determined by expression (14), and the gradient function in the local strength criterion by expressions (15) and (17) with allowance for (16), (21), and (22). The quantity

l_0, which enters into (14), is the critical dimension of a defect shaped like a round crack: $l_0 = \pi K_c^2/(2\sigma_0^2)$. The use of the gradient function (8) in this case can lead to a great error in the determination of the critical dimension of the defect. For example, Nordgren and Melander [16] found the critical dimensions of defects shaped like spherical pores for a WC–10% Co material. The experiments showed that the resulting values differ slightly from the values calculated for a round crack within the framework of the approach of linear fracture mechanics. This means that the coefficient β in expression (14) is zero for the given material, i.e., irrespective of the shape, the defects available in the material do not exert an effect on the strength of the material up to the definite (critical) dimension. The use of the functions (8) in the local strength criterion results, in this case, in an estimate of the critical dimension of the defect that is overestimated by 62% [6].

Thus, the correct application of the gradient approach to estimation of the local strength allows one to obtain physically correct expressions for the ultimate stresses and dimensions of defects that describe well the known experimental data.

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